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Discrete torsion orbifolds and D-branes II

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Abstract: The consistency of the orbifold action on open strings between D-branes in orbifold theories with and without discrete torsion is analysed carefully. For the example of the $\mathbb{C}^3/\mathbb{Z}_2\times\mathbb{Z}_2$ theory, it is found that the consistency of the orbifold action requires that the D-brane spectrum contains branes that give rise to a conventional representation of the orbifold group as well as branes for which the representation is projective. It is also shown how the results generalise to the orbifolds $\mathbb{C}^3/\mathbb{Z}_N \times \mathbb{Z}_N$, for which a number of novel features arise. In particular, the $N > 2$ theories with minimal discrete torsion have non-BPS branes charged under twisted R-R potentials that couple to none of the (known) BPS branes.

Keywords: Superstrings and Heterotic Strings, D-branes, Conformal Field Models in String Theory.

Contents

1. Introduction

One of the ways in which Dirichlet branes have played an important rôle in string theory is that they enable us to obtain insight into the background geometry by analysing the low-energy theory (and in particular its moduli space) of a Dirichlet brane probe. One class of theories for which this is of particular interest are orbifolds with discrete torsion [1] whose geometric interpretation is only partially understood [2]–[5]. The issue of understanding D-branes for this class of theories has attracted some interest recently [6]–[22].

A framework for describing D-branes for general orbifold theories was developed in [23, 24]. In this approach, one begins with an invariant configuration of D-branes on the covering space and restricts the open string spectrum to those states that are invariant under the action of the orbifold group. This 'total' action on the open string states $|\psi, ij\rangle$ can be decomposed into an action on the oscillator state ψ and an action on the Chan-Paton factors ij

$$g|\psi, ij\rangle = \gamma(g)_{ii'} \,|\, \mathrm{U}(g)\psi, i'j'\rangle \, \gamma(g)^{-1}_{j'j} \,. \tag{1.1}$$

It was argued in [23] that the consistency of the group action requires that γ should be a conventional or a projective representation of the orbifold group.

For the case of orbifolds with discrete torsion, Douglas proposed [6] that D-branes are characterised by the property that the representation γ that appears in (1.1) is a *projective* representation of the orbifold group. For the simplest example where we consider the compactification on a torus with a B-field (which induces torsion), this can be intuitively understood as follows. In the presence of a non-trivial B-field, the world-volume theory of a Dirichlet brane is non-commutative [25], and this translates into a 'non-commutative' (i.e. projective) action of the orbifold group on the Chan-Paton factors of the open string.

In general, however, it was argued in [26] that the relation between 'discrete torsion' and 'projective' representations of the orbifold group is more involved. In particular, the specific example of the $\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold with and without discrete torsion was analysed, and the relevant Dirichlet branes were constructed using the boundary state approach. It was found that for branes that are localised at the fixed point, the above relation between discrete torsion and projective representations was satisfied. However, both theories also have branes that carry the other representation (i.e. a projective representation for the theory without discrete torsion, and a conventional representation for the theory with discrete torsion). This was also shown to be necessary in order for the D-brane spectrum to be invariant under T-duality (that relates the theory with and without discrete torsion [2]).

The resulting D-brane spectrum, however, raises a puzzle[1] that we shall resolve in this paper: since each of the two theories has conventional as well as projective Dirichlet branes, the representation γ that appears in (1.1) is a direct sum of a conventional and a projective representation, and therefore neither conventional nor projective. Thus the D-brane spectrum that was found in [26] appears to be in conflict with the results of [23]. As we shall explain in some detail, a careful application of the consistency analysis of [23] actually implies that the theory must have both conventional and projective Dirichlet branes in order to be consistent. This is a consequence of the fact that the action of the orbifold group on the oscillator states is *not* a conventional representation for all open string sectors (as was implicitly assumed in [23]), but rather defines a projective representation for some open string sectors, and a conventional representation for the others.[2]

We shall also analyse the non-BPS D-branes for both theories, paying particular attention to the question of whether the representation of the orbifold group on the Chan-Paton indices is conventional or projective, and how this ties in with the above consistency analysis.

[1]We thank Greg Moore for drawing our attention to this problem.

[2]It was noted in [24], in the context of orientifold theories, that this subtlety may occur.

The $\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold is in many ways special, and it is *a priori* not clear if and in which way the above findings generalise to more general orbifolds. In order to address this issue, we also analyse the case of $\mathbb{C}^3/\mathbb{Z}_N \times \mathbb{Z}_N$ orbifolds without discrete torsion and with minimal discrete torsion in quite some detail. As we shall see, the analysis depends crucially on the value of N, in particular on whether N is odd, twice an odd number, or divisible by four.

One remarkable result is that for $N > 2$ the theory with minimal discrete torsion has non-BPS D-branes that seem to be stable against the decay into BPS brane anti-brane pairs throughout the moduli space. These D-branes carry R-R charges that are not carried by any of the BPS branes of the theory (that have been considered before in [7, 8]). We also find that for N divisible by four, the theory without discrete torsion and the theory with minimal discrete torsion both have fractional D-branes for which the orbifold action on the Chan-Paton factors cannot even be written as a direct sum of conventional and projective representations. These branes are nevertheless (presumably) consistent since the action on the open string oscillator states has the same property.

Throughout the paper, we shall use the boundary state approach for the description and analysis of Dirichlet branes on orbifolds. We shall briefly review some background material in the next subsection, and refer the reader to [8], [21] and [27]–[29], for more details. We shall also briefly summarise some basic facts about discrete torsion; a good introduction can be found in [12] (see also [26, 21]).

The paper is organised as follows. The next subsection contains a brief review of discrete torsion and D-branes on orbifolds. In section 2 we revisit the $\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold with and without discrete torsion. We analyse the consistency of the branes proposed in [26], thereby generalising the framework of [23]. We also discuss some of the non-BPS D-branes in these theories and analyse their consistency. In section 3 we collect some basic facts about the $\mathbb{C}^3/\mathbb{Z}_N \times \mathbb{Z}_N$ orbifolds that shall form the centre of attention for the rest of the paper. The D-brane spectrum of these theories is analysed for odd N in section 4, and for even N in section 5. Section 6 contains some conclusions and open questions.

1.1 Some facts about D-branes on orbifolds and discrete torsion

For our purposes, an orbifold can be thought of as the quotient of a manifold by a discrete group. If the action of the discrete group on the manifold is not free, i.e. if some group elements have fixed points, then the resulting space is singular. An example is the quotient of the real plane $\mathbb{R}^2$ by the $\mathbb{Z}_N$ subgroup of rotations around the origin. In this case the resulting space is a cone with a curvature singularity at the origin. Despite such classical singularities, string theory is well-behaved on orbifolds.

In order to describe in more detail the orbifold construction in string theory, let us consider the example of a closed string theory with background $\mathcal{M}$ on which an (abelian) group G acts as a group of symmetries. The orbifold theory by G consists of those states in the original space of states that are invariant under the action of the orbifold group G. In addition, the theory has so-called twisted sectors containing strings that are closed in $\mathcal{M}/G$ but not in $\mathcal{M}$. If the orbifold action has singularities, the twisted sector states describe degrees of freedom that are localised at the singularities; the presence of these additional states is the essential reason for why string theory is well-behaved despite these singularities.

In the abelian case, there is one twisted sector $\mathcal{H}_h$ for each element $h \in G$. Each twisted sector has to be projected again onto the states that are invariant under the orbifold group G; the corresponding projector is of the form

$$P = \frac{1}{|G|} \sum_{g \in G} g \,, \tag{1.2}$$

and the total partition function of the theory is then

$$Z = \frac{1}{|G|} \sum_{g,h \in G} Z(g,h) \,, \tag{1.3}$$

where

$$Z(g,h) = \mathrm{Tr}_{\mathcal{H}_h}(q^{L_0 - c/24}\, \bar{q}^{\bar{L}_0 - \bar{c}/24} g) \,. \tag{1.4}$$

From a conformal field theory point of view, the presence of the twisted sectors is required by the condition that the total partition function should be modular invariant. However, as was pointed out by Vafa [1], for certain orbifold groups this condition does not uniquely determine the resulting partition function. Indeed, if (1.3) is modular invariant, then so is

$$Z = \frac{1}{|G|} \sum_{g,h \in G} \epsilon(g,h) Z(g,h) \,, \tag{1.5}$$

provided that the phases $\epsilon(g,h)$ satisfy

$$\begin{aligned} \epsilon(h_1 h_2, g) &= \epsilon(h_1, g)\epsilon(h_2, g) \\ \epsilon(h,g) &= \epsilon(g,h)^{-1} \,. \end{aligned} \tag{1.6}$$

The relevant phases $\epsilon(g,h)$ are called discrete torsion phases. The ambiguity that is described by these phases corresponds to an ambiguity in the definition of the orbifold action in each twisted sector.

We now turn to the description of D-branes on orbifolds. For concreteness, let us assume that spacetime is the product of Minkowski space and an orbifold. We first consider a 'bulk' brane that may be extended along some of the directions transverse to the orbifold but that is localised at a generic point in the orbifold. The dynamics of such a D-brane is described in terms of open strings as follows. Consider a preimage of the brane on the covering space and add image branes to obtain a configuration invariant under the orbifold group. (Since we are considering a brane at a generic point of the orbifold, we shall need a total of $|G|$ copies.) Then consider all open strings with endpoints on any (two) of these branes. The excitations of the D-brane are described by the open string states that are invariant under the action of G. As mentioned before, the action of a group element on the open string states can be written in terms of an action on the Chan-Paton indices and an action on the oscillator states (see (1.1)). In the case of a bulk brane the representation on the Chan-Paton indices γ is the regular representation of G (which has indeed dimension $|G|$).

If the D-brane is localised at a singular point of the orbifold, the dimension of γ may be smaller. This is a consequence of the fact that we need fewer preimages in the covering space to make an orbifold invariant configuration. Branes for which the dimension of γ is strictly smaller than $|G|$ are called 'fractional' D-branes. Because the dimension of γ is smaller than that of a bulk D-brane, fractional D-branes cannot move off the singular point; instead, a number of fractional D-branes have to come together in order for the system to be able to move off into the bulk.

D-branes describe open string sectors that can be added consistently to a given closed string theory. In order to analyse this consistency condition, it is often useful to consider an annulus (or cylinder) diagram for which the boundary conditions are determined by two (possibly identical) D-branes — one for each boundary. In the simplest case we have a diagram without an insertion of a vertex operator. This diagram can be given two different hamiltonian interpretations, depending on which world-sheet coordinate is chosen as the world-sheet time. From an open string point of view, the diagram is interpreted as a one-loop vacuum diagram. In an orbifold theory, this diagram will always contain a projector (1.2) that ensures that only orbifold invariant open string states run in the loop. On the other hand, from the closed string point of view the diagram describes the tree-level exchange of a closed string between two sources (D-branes). Each D-brane can then be described by a 'boundary state' $|D\rangle$, a coherent state in the closed string theory that describes the emission and absorption of closed string states by the D-brane. The condition that both the open and the closed string interpretations of the annulus diagram should be sensible imposes strong restrictions on the possible D-branes in a given closed string theory.

Let us close this brief review by summarising some of the most important features of the boundary state construction for orbifold theories. (More details can be found in [8, 21] and [27]–[29].) In an orbifold theory a boundary state is typically a sum of components that are defined in each untwisted and twisted sector of the theory. The component in a given sector describes the coupling of the D-brane to closed strings in that sector. Bulk branes are described by boundary states whose only non-vanishing components are in the untwisted sectors. On the other hand, the boundary state of a fractional brane has generically a non-trivial component in at least one twisted sector. The contribution of the h-twisted sector to the cylinder diagram considered before corresponds, from the open string point of view, to the one-loop diagram with the insertion of h. The sum over twisted sectors reproduces then the projection operator (1.2). In particular, boundary states with a non-trivial component in the h-twisted sector lead to open strings for which the annulus diagram gets a non-trivial contribution from the insertion of h. This implies that the world-volume of the corresponding D-brane must intersect its image under the action of h.

2. The $\mathbb{Z}_2 \times \mathbb{Z}_2$ case.

Let us begin by reviewing the case of the $\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold with and without discrete torsion. The following discussion extends the results and the consistency analysis of [26]. For simplicity we shall consider the uncompactified theory.

The orbifold group is generated by g_1 and g_2 where $g_1^2 = g_2^2 = 1$ and $g_1 g_2 = g_2 g_1$. These generators act by inversion on some of the coordinates $x^3, \ldots, x^8$. More specifically, g_1 maps $x^i \mapsto -x^i$ for $i = 5, 6, 7, 8$ and g_2 maps $x^j \mapsto -x^j$ for $j = 3, 4, 7, 8$.

The second cohomology group $H^2(\mathbb{Z}_2 \times \mathbb{Z}_2, \mathrm{U}(1))$ is $\mathbb{Z}_2$, and there are therefore two orbifold theories: the theory without discrete torsion and the theory with discrete torsion, for which g_i acts in the g_j-twisted sector (where $i \neq j$) with a relative minus sign. In order to describe the D-brane spectrum of these theories, it is convenient to introduce the following notation: we denote, as in [28], a Dirichlet p-brane by $(r; s_1, s_2, s_3)$ where $p = r + s_1 + s_2 + s_3$, provided that it has $r+1$ Neumann boundary conditions along the directions that have not been affected by the orbifold, i.e. x^0, x^1, x^2, x^9, and s_i Neumann boundary conditions along the directions x^{2i+1} and x^{2i+2}. We shall always fix x^0 and x^9 to be the light-cone coordinates; x^1 and x^2 are unaffected by the orbifold.

In the following we shall describe both type IIA and type IIB in a uniform fashion. Most of the analysis will be the same for both cases, the only difference being the possible values of r for a given choice of s_i. Unless specified otherwise, we will always assume that $p = r + s_1 + s_2 + s_3$ is even in IIA and odd in IIB.

It is well known that D-branes couple to R-R potentials. It is therefore worthwhile to summarise the spectrum of R-R ground states of the orbifolds we are studying by giving their Hodge diamonds. In the theory without discrete torsion, the untwisted sector contributes [2]

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 3 & & 0 & \\ 1 & & 3 & & 3 & & 1 \\ & 0 & & 3 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array} \tag{2.1}$$

to the Hodge diamond. The total contribution of the three twisted sectors is

$$\begin{array}{ccccccc} & & & 0 & & & \\ & & 0 & & 0 & & \\ & 0 & & 3 & & 0 & \\ 0 & & 0 & & 0 & & 0\,. \\ & 0 & & 3 & & 0 & \\ & & 0 & & 0 & & \\ & & & 0 & & & \end{array} \tag{2.2}$$

In the theory with discrete torsion, the untwisted contribution is the same, while the twisted sectors now contribute

$$\begin{array}{ccccccc} & & & 0 & & & \\ & & 0 & & 0 & & \\ & 0 & & 0 & & 0 & \\ 0 & & 3 & & 3 & & 0\,. \\ & 0 & & 0 & & 0 & \\ & & 0 & & 0 & & \\ & & & 0 & & & \end{array} \tag{2.3}$$

If we were to compactify the theory, obtaining $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds, there would be more fixed points and correspondingly larger contributions from the twisted sectors. For most of our analysis it will however be sufficient to consider the non-compact situation.

2.1 The theory without discrete torsion

The theory without discrete torsion has conventional fractional Dirichlet branes for which all s_i are even. For the simplest case where $s_i = 0$, this brane is stuck at the fixed plane of the orbifold group, $x^3 = \cdots = x^8 = 0$. All these branes are described by a superposition of boundary states where we have a non-trivial component in every closed string sector of the theory. These components are invariant under the GSO- and the orbifold projection, and the branes are charged with respect to the twisted and untwisted R-R potentials.

In addition to this conventional fractional Dirichlet brane, the theory also has supersymmetric 'projective fractional' D$(r;1,1,1)$ branes. For a fixed orientation, the moduli space of the brane consists of three branches, namely the fixed planes of g_1, g_2 and $g_3 = g_1 g_2$. The boundary state description of the brane is slightly different for the different branches of the moduli space, and we shall in the following only give the explicit formula for the case of the g_1 branch. (The relevant modifications for the other branches are self-evident.) Let us denote by $\mathbf{y}$ the position of the brane in the directions that are unaffected by the orbifold action, and by $\mathbf{a}$ the coordinates in the x^3, x^4 directions on the fixed plane of g_1. The relevant boundary state is then of the form

$$\begin{aligned}
|D(r;s_1,s_2,s_3);\mathbf{y},\mathbf{a},\epsilon,\epsilon'\rangle = \; & |D(r;s_1,s_2,s_3);\mathbf{y},\mathbf{a}\rangle_{\mathrm{NS-NS;U}} + \\
& +\epsilon|D(r;s_1,s_2,s_3);\mathbf{y},\mathbf{a}\rangle_{\mathrm{R-R;U}} + \\
& +\epsilon'\Big(|D(r;s_1,s_2,s_3);\mathbf{y},\mathbf{a}\rangle_{\mathrm{NS-NS;T}_{g_1}} + \\
& \qquad + \epsilon|D(r;s_1,s_2,s_3);\mathbf{y},\mathbf{a}\rangle_{\mathrm{R-R;T}_{g_1}}\Big) + \\
& +|D(r;s_1,s_2,s_3);\mathbf{y},-\mathbf{a}\rangle_{\mathrm{NS-NS;U}} + \\
& +\epsilon|D(r;s_1,s_2,s_3);\mathbf{y},-\mathbf{a}\rangle_{\mathrm{R-R;U}} - \\
& -\epsilon'\Big(|D(r;s_1,s_2,s_3);\mathbf{y},-\mathbf{a}\rangle_{\mathrm{NS-NS;T}_{g_1}} + \\
& \qquad + \epsilon|D(r;s_1,s_2,s_3);\mathbf{y},-\mathbf{a}\rangle_{\mathrm{R-R;T}_{g_1}}\Big)\,,
\end{aligned} \tag{2.4}$$

where $s_1 = s_2 = s_3 = 1$ and $\epsilon, \epsilon' = \pm 1$. Here and in the following we always restrict $\mathbf{a}$ to parametrise the 'reduced' space; in the present case this means that $\mathbf{a}$ parametrises for example the half-space characterised by $a_3 \geq 0$. It was shown in [26] that this state is invariant under the GSO- and orbifold projection, and that it gives rise to the projective representation of the orbifold group

$$\begin{aligned}
\gamma(g_1) &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\
\gamma(g_2) &= \begin{pmatrix} 0 & \pm 1 \\ 1 & 0 \end{pmatrix} \\
\gamma(g_3) &= \begin{pmatrix} 0 & \mp 1 \\ 1 & 0 \end{pmatrix} \\
\gamma((-1)^F) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\end{aligned} \tag{2.5}$$

on the 2×2 Chan-Paton indices of the open strings that begin and end on the $\mathrm{D}(r;1,1,1)$ brane.

While the boundary state contains components from the g_1-twisted sector, it is not charged with respect to any massless R-R field in the g_1-twisted sector; this is simply a consequence of the fact that the massless ground state of the boundary state (with zero momentum) is independent of $\mathbf{a}$, and that the above boundary state consists of the difference between the state at $\mathbf{a}$ and the one at $-\mathbf{a}$. However, the above boundary states are charged under the massless R-R fields in the untwisted sector. In fact, there are eight different such massless fields that correspond to the eight different orientations of the $\mathrm{D}(r;1,1,1)$ brane along the internal directions; all of these D-branes are consistent.

One may wonder whether it is consistent to have both a conventional fractional brane and a projective fractional brane in one theory. In particular, one may ask whether the action of the orbifold group on the open string that begins on the conventional fractional brane and ends on the projective fractional brane respects the relations of the orbifold group.[3] As we shall see momentarily, this is indeed consistent. Actually, one could have turned the argument around and predicted that the $D(r;1,1,1)$ brane has to carry a projective representation of the orbifold group from the consistency analysis of this open string.

The open string between the conventional fractional brane and the projective fractional brane has two Chan-Paton indices that label on which of the two $D(r;1,1,1)$ branes the open string begins (or ends). Under the action of the orbifold group, this 2-vector transforms in the projective representation of the orbifold group that characterises the $D(r;1,1,1)$ brane. In order for the whole action of the orbifold group to be consistent we therefore have to have that the action on the oscillator states of the corresponding open string is also projective. For any pair of a conventional fractional and a projective fractional brane, the open string has three fermionic zero modes in the internal directions, and an odd number of fermionic zero modes in the directions that are unaffected by the action of the orbifold group. (This is true for both the NS and the R sector.) Of the three fermionic zero modes in the internal directions, one is always in the x^3-x^4 plane, one in the x^5-x^6 plane and one in the x^7-x^8 plane; for definiteness, let us assume (without loss of generality) that the relevant fermionic zero modes are ψ_0^3, ψ_0^5 and ψ_0^7. The action of g_1 and g_2 on the ground states of the open string is then given by

$$\begin{aligned} g_1 &= \pm 2i\psi_0^5\psi_0^7 \\ g_2 &= \pm 2i\psi_0^3\psi_0^7\,, \end{aligned} \tag{2.6}$$

where the prefactor has been fixed (up to a sign) by the condition that $g_1^2=g_2^2=1$. (We are assuming here that the fermionic zero modes satisfy the Clifford algebra $\{\psi_0^\mu,\psi_0^\nu\}=\delta^{\mu,\nu}$.) Irrespective of the choices for the signs, we then have the identity

$$g_1 g_2 = -g_2 g_1\,, \tag{2.7}$$

and this implies that the action of the orbifold group on the oscillator states of the open string is also projective. Taken together with the projective representation on the Chan-Paton indices, the whole action is then a conventional representation, as has to be the case for consistency. Thus we have seen that it is necessary for consistency that the $D(r;1,1,1)$ has a projective representation of the orbifold group.

It may be worthwhile to point out that the above D-brane spectrum falls slightly outside the framework described in [23]. As we mentioned in the introduction, the action of the orbifold group on the open string space of states can be written as

$$g|\psi,ij\rangle = \gamma(g)_{ii'}\,|\,\mathrm{U}(g)\psi,i'j'\rangle\,\gamma(g)^{-1}_{j'j}\,, \tag{2.8}$$

[3]This issue was not analysed in [26]. We thank Greg Moore for drawing our attention to this point.

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where ψ is an element of the open string Hilbert space, $\psi \in \mathcal{H}_{ij}$ (that depends in general on the Chan-Paton indices ij), the action on the Chan-Paton indices is given by the representation γ while that on the open string states in $\mathcal{H} = \oplus_{ij}\mathcal{H}_{ij}$ is described by U. If we assume (as was done in [23]) that U defines a conventional representation of the orbifold group, it then follows (using the factorisation properties of the open string diagrams) that γ must define a conventional or a projective representation of the orbifold group. What we have found above is that γ is a direct sum of projective and conventional representations of the orbifold group (and therefore *neither* projective nor conventional). This is consistent with the arguments of [23] because U does not define a conventional representation of the orbifold group on the whole space of open string states. (Indeed, as we have just shown, U acts for example projectively in the sector of the open strings between $\mathrm{D}(r;0,0,0)$ and $\mathrm{D}(r';1,1,1)$, while it describes a conventional representation in all sectors describing strings that begin and end on the same brane.) In essence, the consistency analysis of [23] amounts to checking that the total orbifold action defines a conventional representation for every open string sector; this is what we have verified above, and what we shall analyse in the following.

It is not difficult to see that the above branes are the only supersymmetric branes of the theory and that they account already for all R-R charges of the theory. In addition to these BPS branes, the theory also has a number of non-BPS branes. One of them is yet another kind of projective fractional brane for which one of the three s_i is even, while the other two are odd. This brane has been discussed before in [30], but it was not realised there that it gives rise to a projective representation of the orbifold group (as we shall see momentarily). For definiteness, let us assume that s_1 is even; the boundary state of this brane is then given by

$$\begin{aligned}
|D(r;s_1,s_2,s_3);\mathbf{y},\mathbf{a},\epsilon,\epsilon'\rangle = \; & |D(r;s_1,s_2,s_3);\mathbf{y},\mathbf{a}\rangle_{\mathrm{NS-NS;U}} + \\
& +\epsilon|D(r;s_1,s_2,s_3);\mathbf{y},\mathbf{a}\rangle_{\mathrm{R-R;U}} + \\
& +\epsilon'\Big(|D(r;s_1,s_2,s_3);\mathbf{y},\mathbf{a}\rangle_{\mathrm{NS-NS;T}_{g_1}} + \\
& \qquad + \epsilon|D(r;s_1,s_2,s_3);\mathbf{y},\mathbf{a}\rangle_{\mathrm{R-R;T}_{g_1}}\Big) + \\
& +|D(r;s_1,s_2,s_3);\mathbf{y},-\mathbf{a}\rangle_{\mathrm{NS-NS;U}} - \\
& -\epsilon|D(r;s_1,s_2,s_3);\mathbf{y},-\mathbf{a}\rangle_{\mathrm{R-R;U}} - \\
& -\epsilon'\Big(|D(r;s_1,s_2,s_3);\mathbf{y},-\mathbf{a}\rangle_{\mathrm{NS-NS;T}_{g_1}} - \\
& \qquad - \epsilon|D(r;s_1,s_2,s_3);\mathbf{y},-\mathbf{a}\rangle_{\mathrm{R-R;T}_{g_1}}\Big)\,, \qquad (2.9)
\end{aligned}$$

where now s_1 is even and $s_2 = s_3 = 1$. It is easy to see from of [26, table 2] that this state is invariant under the action of the GSO projection and the orbifold group. The corresponding D-brane is charged under a massless R-R field from the g_1-twisted sector. It is also clear that this boundary state reduces to the expression given in [30] as $\mathbf{a} \to 0$.

The determination of the corresponding projection operators in the open string requires a little bit of care. First of all, since the two copies of the brane (at **a** and $-\mathbf{a}$) have opposite bulk R-R charge, the action of $(-1)^F$ in the open string involves a non-trivial action on the Chan-Paton factors which is given by (conjugation with)

$$\gamma\Big((-1)^F\Big) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{2.10}$$

In addition, the action of the orbifold group on the Chan-Paton indices is given by the matrices

$$\begin{aligned} \gamma(g_1) &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ \gamma(g_2) &= \begin{pmatrix} 0 & \pm 1 \\ 1 & 0 \end{pmatrix} \\ \gamma(g_3) &= \begin{pmatrix} 0 & \mp 1 \\ 1 & 0 \end{pmatrix} , \end{aligned} \tag{2.11}$$

as can be read off from the boundary state (this is described in some detail in [26]). As before, this is a projective representation of the orbifold group, but now also the action of $(-1)^F$ does not commute any more with the action of g_2 and g_3: in fact we have

$$\begin{aligned} \gamma(g_1)\gamma\Big((-1)^F\Big) &= \gamma\Big((-1)^F\Big)\gamma(g_1) \\ \gamma(g_2)\gamma\Big((-1)^F\Big) &= -\gamma\Big((-1)^F\Big)\gamma(g_2) \\ \gamma(g_3)\gamma\Big((-1)^F\Big) &= -\gamma\Big((-1)^F\Big)\gamma(g_3) . \end{aligned} \tag{2.12}$$

As before, we need to check whether these non-BPS branes are mutually consistent with the BPS branes that we have described above, i.e. whether the total action on the open strings between a BPS brane and a non-BPS brane respects the relations of the orbifold group and of $(-1)^F$. It is not difficult to see that this is indeed the case. Consider, for instance, the NS sector of the open string between a projective fractional D$(r;0,1,1)$ brane extended along the x^5 and x^7 directions and a conventional fractional D$(r';0,0,0)$ brane. The fermion zero modes in the orbifold directions are ψ_0^5 and ψ_0^7, and the action of the group elements on the ground states of the open string is then given by

$$\begin{aligned} g_1 &= \pm 2i\psi_0^5\psi_0^7 \\ g_2 &= \pm\sqrt{2}\psi_0^7\Gamma \\ g_3 &= \pm\sqrt{2}\psi_0^5\Gamma , \end{aligned} \tag{2.13}$$

where Γ is the chirality operator that is proportional to the product of all fermionic zero modes. The total number of fermionic zero modes is even and therefore Γ commutes with $(-1)^F$. This implies that both g_2 and g_3 anti-commute with $(-1)^F$, thus providing precisely the signs that cancel the signs in (2.12). We also have that the orbifold operators among themselves satisfy the relations of the projective representation of the orbifold group; if we combine this action with the projective action on the Chan-Paton factors (2.11) we get a conventional representation of the orbifold group, as required by consistency. The analysis for the R sector is analogous.

The situation for the string between the non-BPS brane and the projective fractional $\mathrm{D}(r';1,1,1)$ brane is similar. Again, if we consider the NS sector of the open string between a projective fractional $\mathrm{D}(r;0,1,1)$ brane along the x^5 and x^7 directions and a projective fractional $\mathrm{D}(r';1,1,1)$ brane along the x^3, x^5 and x^7 directions, we have only one fermionic zero mode, namely ψ_0^3. The group elements are then represented on the ground states of the open string by

$$\begin{aligned} g_1 &= 1 \\ g_2 &= \pm\sqrt{2}\psi_0^3\Gamma \\ g_3 &= \pm\sqrt{2}\psi_0^3\Gamma\,, \end{aligned} \tag{2.14}$$

where Γ is again the chirality operator. Again both g_2 and g_3 anti-commute with $(-1)^F$, but now the orbifold generators commute among themselves and therefore define a conventional representation of the orbifold group. The Chan-Paton indices of both branes transform in the same (projective) representation of the orbifold group, and thus the action on the Chan-Paton indices is a conventional representation of the orbifold group. Together with the above (conventionial) representation on the oscillator states, the total action on the open string states therefore satisfies the relations of the orbifold group. The same also applies for the commutation relations between $(-1)^F$ and the orbifold generators.

There is yet another consistency condition that we should analyse, but that is somewhat more subtle. In the above, we have only analysed the $\mathrm{D}(r;0,1,1)$ brane (or the $\mathrm{D}(r;2,1,1)$ brane), but we have implicitly claimed that the theory also has branes where s_2 or s_3 is even while the other two s_j are odd. In each case, we can repeat the above analysis and thereby demonstrate that each of these branes gives rise to a consistent string between the non-BPS brane in question and any BPS brane. However, we also need to check whether the open string between two different such non-BPS branes is consistent. As an example, let us consider the open string between the $\mathrm{D}(r;0,1,1)$ brane and the $\mathrm{D}(r;1,0,1)$ brane. The action of the orbifold group on the Chan-Paton indices of the two branes is the same, giving rise to a conventional representation of the orbifold group on the Chan-Paton factors; in order for the whole open string to be consistent we therefore have to have that the action on the oscillator states is also a conventional representation of the orbifold group.

Unfortunately, the precise action of the orbifold group on the open string oscillator states is not straightforward to determine, and we are therefore not able to check this consistency condition directly. The difficulty in establishing the orbifold action on the open string states is due to the fact that the moduli spaces of the two branes in question are different: the moduli space of the $D(r;0,1,1)$ brane is the fixed plane of g_1, while that of the $D(r;1,0,1)$ is the fixed plane of g_2. Under the action of g_1 the open string that begins at the $D(r;0,1,1)$ localised at $\mathbf{a} = (a_3, a_4)$ and ends at the $D(r;1,0,1)$ localised at $\mathbf{b} = (b_5, b_6)$ is mapped to a string that begins at (a_3, a_4) and ends at $(-b_5, -b_6)$. Since these two strings are not parallel to one another, there is no canonical way in which we can identify the corresponding Hilbert spaces, and therefore no sense in which g_1 acts as a product of fermionic zero modes. The situation is similar for the action of g_2.

It may also be worthwhile to point out that if it *was* possible to define the action of g_i in terms of fermionic zero modes, our analysis would imply that the $D(r;0,1,1)$ and the $D(r;1,0,1)$ are mutually inconsistent (which would seem somewhat unlikely). Indeed, the R-sector of the open string between these two branes has (for a suitable orientation of the branes) the zero modes ψ_0^4 and ψ_0^6. If we could define $g_1 = \sqrt{2}\psi_0^6\Gamma$ and $g_2 = \sqrt{2}\psi_0^4\Gamma$, we would have $g_1 g_2 = -g_2 g_1$, and the total orbifold action on the open string states would be inconsistent.

In the uncompactified theory, the non-BPS $D(r;0,1,1)$ brane is unstable, but it becomes stable if we compactify the orbifolded directions along which the brane wraps (x^5 and x^7, say) on sufficiently small circles [30]. Indeed, it is clear from the boundary state (2.9) that the string between the brane at $\mathbf{a}$ and the brane at $-\mathbf{a}$ (these strings correspond to the off-diagonal Chan-Paton indices) has the 'wrong' GSO-projection, and therefore that the tachyonic ground state survives the GSO-projection. However, it is also clear from (2.11) that the orbifold projection acts as $(1-g_1)/2$ on these open strings, and therefore that the ground state tachyon is not orbifold invariant. On the other hand, the open string state with momentum p^5 is mapped to the state with momentum $-p^5$ under the action of g_1, and therefore the anti-symmetric combination of these two states is invariant under g_1. Furthermore, by considering a suitable linear combination of the two off-diagonal Chan-Paton indices, the state can be made invariant under the action of g_2, and therefore under the action of the whole orbifold group. The resulting physical state is tachyonic provided that R^5 is sufficiently large and the separation between both copies of the brane is sufficiently small — for instance, for $\mathbf{a} = 0$ the precise condition on R^5 is $R^5 > \sqrt{2\alpha'}$; the non-BPS D-brane can therefore only be stable if R^5 is sufficiently small (i.e. $R^5 < \sqrt{2\alpha'}$). In the regime where the $D(r;0,1,1)$ brane is unstable it decays presumably into two non-BPS $D(r;0,0,1)$ branes of the type considered in [30],[4] or into four fractional $D(r;0,0,0)$ branes, depending on R^7. Other instabilities arise if the 6 and 8 directions are compactified on sufficiently small circles because we obtain then tachyonic winding states.

[4] It may be worth pointing out that for the $D(r;0,0,1)$ branes of [30] $r+1$ is odd in IIA and even in IIB; thus these branes occur for the *same* value of r as the non-BPS $D(r;0,1,1)$ brane we have just discussed.

The non-BPS $\mathrm{D}(r';0,0,1)$ brane discussed in [30] is quite special to $N=2$, and does not generalise to $N>2$. However, the theory has yet another non-BPS $\mathrm{D}(r;0,0,1)$ brane (where now $r+1$ is even in IIA and odd in IIB) which will generalise to $N>2$. If we require that the open strings between this brane and the BPS branes are consistent, an analogous analysis to the above implies that the action of the orbifold group on the Chan-Paton indices must be a conventional representation, but that $(-1)^F$ must act non-trivially so that it anticommutes with g_1 and g_2 (but not g_3). A boundary state that gives rise to this action is of the form

$$\begin{aligned}
|D(r;s_1,s_2,s_3);\mathbf{y},\mathbf{c},\epsilon,\epsilon'\rangle = {} & |D(r;s_1,s_2,s_3);\mathbf{y},\mathbf{c}\rangle_{\mathrm{NS-NS;U}} + \\
& +\epsilon|D(r;s_1,s_2,s_3);\mathbf{y},\mathbf{c}\rangle_{\mathrm{R-R;U}} + \\
& +\epsilon'\Big(|D(r;s_1,s_2,s_3);\mathbf{y},\mathbf{c}\rangle_{\mathrm{NS-NS;T}_{g_3}} + \\
& \qquad + \epsilon|D(r;s_1,s_2,s_3);\mathbf{y},\mathbf{c}\rangle_{\mathrm{R-R;T}_{g_3}}\Big) + \\
& +|D(r;s_1,s_2,s_3);\mathbf{y},-\mathbf{c}\rangle_{\mathrm{NS-NS;U}} - \\
& -\epsilon|D(r;s_1,s_2,s_3);\mathbf{y},-\mathbf{c}\rangle_{\mathrm{R-R;U}} \\
& +\epsilon'\Big(|D(r;s_1,s_2,s_3);\mathbf{y},-\mathbf{c}\rangle_{\mathrm{NS-NS;T}_{g_3}} - \\
& \qquad - \epsilon|D(r;s_1,s_2,s_3);\mathbf{y},-\mathbf{c}\rangle_{\mathrm{R-R;T}_{g_3}}\Big) ,
\end{aligned} \tag{2.15}$$

where $s_1=s_2=0$, $s_3=1$ and $\mathbf{c}$ parametrises now the fixed plane of g_3. It is again easy to see that this boundary state is invariant under the GSO- and the orbifold projection. However, because of the relative minus signs, the resulting D-brane is uncharged with respect to any R-R potential.

As before, we can read off from the above boundary state the action of the various operators on the Chan-Paton indices

$$\begin{aligned}
\gamma(g_1) &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
\gamma(g_2) &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
\gamma(g_3) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
\gamma((-1)^F) &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .
\end{aligned} \tag{2.16}$$

This defines indeed a conventional representation of the orbifold group for which $(-1)^F$ anti-commutes with g_1 and g_2.

Incidentally, a similar consistency analysis also applies to the brane proposed in [30]. The main difference to the situation above is that the overall number of fermionic zero modes for the strings between the non-BPS brane and the BPS branes is odd in that case. In order to be able to define the chirality operator Γ (that enters

	BPS/non-BPS	type of representation
D(r;even,even,even)	BPS	conventional
D(r;1,1,1)	BPS	projective (2.5)
D(r;even,1,1)	non-BPS	projective (2.10) and (2.11)
D(r;even,even,1)	non-BPS	'conventional' (2.16)

Table 1: The D-brane spectrum of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold without discrete torsion.

in the definition of the orbifold operators as in (2.13) and (2.14) it is then necessary to introduce an additional 'boundary' fermion, as discussed in a similar situation by Witten [31]. This procedure doubles the degrees of freedom of the open string, and allows for an action of the matrices (2.16) on the (two-dimensional) space of multiplicities. The resulting open string loop amplitudes are then in agreement with those that follow from the boundary state given in [30].

We also have to check that the non-BPS branes for which one s_i is odd (while the other two s_j are even) are consistent with themselves and the other non-BPS branes. In those cases where the relevant branes are defined on the same branch, the orbifold generators can be expressed in terms of fermionic zero modes, and we have verified, using similar arguments as above, that the branes are indeed consistent. In the other cases, the situation is more complicated, and we do not know how to check this consistency condition directly.

Finally, it also follows from (2.16) that the above non-BPS $\mathrm{D}(r;0,0,1)$ brane is unstable, irrespective of whether we compactify or not. Indeed, the off-diagonal components of the Chan-Paton matrix correspond to strings with the wrong GSO-projection, and therefore contain a tachyonic ground state in the NS sector. Since g_3 acts as the identity matrix, both these states are invariant under g_3; a certain linear combination of them is thus invariant under g_1 and g_2, and therefore defines a physical tachyonic state in the open string.

For the convenience of the reader, let us summarise in table 1 the D-branes whose boundary states we have discussed above. In this table it is understood that all permutations of 'even' and 1 are included, and that r is determined in terms of s_i as discussed before. In the last case, the action of the orbifold group is a conventional representation, but it does not commute with $(-1)^F$.

2.2 The theory with discrete torsion

For the theory with discrete torsion, the rôles of the conventional and projective fractional branes are reversed. For s_i even, the theory now has a projective fractional brane. Again, its moduli space consists of the three fixed planes of g_1, g_2 and $g_3 = g_1 g_2$. In the g_1-branch the relevant boundary state is then given by (2.4) where now s_i is even. As before, this state is invariant under the GSO- and orbifold projection,

	BPS/non-BPS	type of representation
D(r;even,even,even)	BPS	projective (2.5)
D(r;1,1,1)	BPS	conventional
D(r;even,1,1)	non-BPS	'conventional' (2.16)
D(r;even,even,1)	non-BPS	projective (2.10) and (2.11)

Table 2: The D-brane spectrum of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold with discrete torsion.

and it gives rise to a projective representation of the orbifold group on the 2×2 Chan-Paton indices of the open strings that begin and end on this brane.

In the limit $\mathbf{a} \to 0$, this boundary state reduces to what has been described before in [8] (see also [21]), but the above description is more general, and in particular describes the relevant boundary state for all points on its moduli space. The fact that the boundary state involves components from the g_1-twisted sector suggests that the brane cannot move off the fixed planes that describe its moduli space.

The theory also has a conventional fractional D$(r;1,1,1)$ brane. The corresponding boundary state has components in all closed string sectors of the theory; it is given by

$$\begin{aligned}
|D(r;1,1,1);\mathbf{y},\epsilon,\epsilon_1,\epsilon_2\rangle = {} & |D(r;1,1,1);\mathbf{y}\rangle_{\mathrm{NS-NS;U}} + \epsilon|D(r;1,1,1);\mathbf{y}\rangle_{\mathrm{R-R;U}} + \\
& +\epsilon_1\Big(|D(r;1,1,1);\mathbf{y}\rangle_{\mathrm{NS-NS;T}_{g_1}} + \\
& \qquad + \epsilon|D(r;1,1,1);\mathbf{y}\rangle_{\mathrm{R-R;T}_{g_1}}\Big) \\
& +\epsilon_2\Big(|D(r;1,1,1);\mathbf{y}\rangle_{\mathrm{NS-NS;T}_{g_2}} + \\
& \qquad + \epsilon|D(r;1,1,1);\mathbf{y}\rangle_{\mathrm{R-R;T}_{g_2}}\Big) + \\
& +\epsilon_1\epsilon_2\Big(|D(r;1,1,1);\mathbf{y}\rangle_{\mathrm{NS-NS;T}_{g_3}} + \\
& \qquad + \epsilon|D(r;1,1,1);\mathbf{y}\rangle_{\mathrm{R-R;T}_{g_3}}\Big) \,. \qquad (2.17)
\end{aligned}$$

Because of the same argument as above, the open string between the two types of branes is then again consistent. Also, these are the only supersymmetric branes, and they account for all R-R charges of the theory.

There is now a consistent D-brane for which one of the s_i is odd, while the other two are even. If $s_1 = 1$, the relevant boundary state is described by (2.9). This brane carries twisted R-R charge and is stable (for a certain regime of radii in the compactified theory) but non-BPS. Finally, the theory has an unstable, uncharged, non-BPS D-brane for which precisely one s_i is even. The consistency and stability analysis is as in the case without discrete torsion. All of this is in agreement with T-duality that relates the theory with and without discrete torsion [2]. The D-brane spectrum can be summarised by table 2.

The main aim of this paper is to explain how the above analysis generalises to the case of a general $\mathbb{Z}_N \times \mathbb{Z}_N$ orbifold. In doing so, we shall encounter a number of interesting and new phenomena. As shall become apparent, the analysis depends on whether N is odd or even, and further on whether an even N is divisible by four or not. After mentioning some generalities, we shall discuss these different cases separately in the following.

3. The $\mathbb{Z}_N \times \mathbb{Z}_N$ orbifold: generalities

Let us consider the $\mathbb{Z}_N \times \mathbb{Z}_N$ orbifold of $\mathbb{R}^6$. We identify $\mathbb{R}^6 \simeq \mathbb{C}^3$ by defining

$$\begin{aligned} z_1 &= x^3 + ix^4 \\ z_2 &= x^5 + ix^6 \\ z_3 &= x^7 + ix^8 \,. \end{aligned} \tag{3.1}$$

The two cyclic groups then act on z_i by

$$\begin{aligned} g_1(z_1, z_2, z_3) &= (z_1, e^{-2\pi i/N} z_2, e^{2\pi i/N} z_3) \\ g_2(z_1, z_2, z_3) &= (e^{2\pi i/N} z_1, z_2, e^{-2\pi i/N} z_3) \,. \end{aligned} \tag{3.2}$$

The possible discrete torsion theories are classified by $H^2(\mathbb{Z}_N \times \mathbb{Z}_N; U(1)) = \mathbb{Z}_N$. Indeed, the possible discrete torsion phases $\epsilon(g, h)$ are determined in terms of

$$\omega \equiv \epsilon(g_1, g_2) = e^{2\pi i m/N} \qquad \text{where } m = 0, \ldots, N-1 \,. \tag{3.3}$$

This fixes all the phases $\epsilon(g, h)$ since we have the relations

$$\begin{aligned} \epsilon(h_1 h_2, k) &= \epsilon(h_1, k)\epsilon(h_2, k) \\ \epsilon(h, g) &= \epsilon(g, h)^{-1} \,. \end{aligned} \tag{3.4}$$

In this paper, we shall only consider 'minimal' discrete torsion, which means that ω is a generator of $\mathbb{Z}_N$.

Let us also give the Hodge diamond that summarises the spectrum of R-R ground states. In the theory without discrete torsion, the untwisted sector contributes (for $N > 2$) [7]

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 3 & & 0 & \\ 1 & & 0 & & 0 & & 1 \\ & 0 & & 3 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array} \tag{3.5}$$

to the Hodge diamond. The contribution of the twisted sectors is

$$
\begin{array}{ccccccc}
 & & & 0 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & \frac{(N+4)(N-1)}{2} & & 0 & \\
0 & & 0 & & 0 & & 0\,. \\
 & 0 & & \frac{(N+4)(N-1)}{2} & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 0 & & &
\end{array}
\tag{3.6}
$$

In the theory with minimal discrete torsion, the untwisted contribution is the same, while the twisted sectors now contribute

$$
\begin{array}{ccccccc}
 & & & 0 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 0 & & 0 & \\
0 & & 3 & & 3 & & 0\,. \\
 & 0 & & 0 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 0 & & &
\end{array}
\tag{3.7}
$$

The BPS D-branes that have been constructed in the literature [7, 8] do not couple to these twisted R-R potentials; it is one of the aims of this paper to construct D-branes that carry these charges.

4. The $\mathbb{Z}_N \times \mathbb{Z}_N$ orbifold for odd N

4.1 The theory without discrete torsion

First of all, as in the $N = 2$ case studied in section 2, the theory has conventional fractional BPS D-branes for which all s_i are even. These branes are charged under twisted and untwisted R-R potentials. The corresponding boundary states are given by

$$
\begin{aligned}
&|D(r; s_1, s_2, s_3); \mathbf{y}, \epsilon, \epsilon_1, \hat{\epsilon}_1\rangle = \\
&= \sum_{m,n=0}^{N-1} \epsilon_1^m \hat{\epsilon}_1^n \Big(|D(r; s_1, s_2, s_3); \mathbf{y}, \mathbf{0}\rangle_{\mathrm{NS-NS;T}_{g_1^m g_2^n}} + \epsilon |D(r; s_1, s_2, s_3); \mathbf{y}, \mathbf{0}\rangle_{\mathrm{R-R;T}_{g_1^m g_2^n}} \Big)\,,
\end{aligned}
\tag{4.1}
$$

where, for notational convenience, we have considered the untwisted sector to be the sector twisted by the trivial group element.

For the remaining branes, there are substantial differences compared to the $N = 2$ case. In particular, when some of the s_i are odd, the orbifold group maps a copy of the brane in the covering space to other copies with different orientations.

For instance, to build an invariant configuration of $D(r;1,1,1)$ branes in the covering space, we now need at least N^2 copies, which can therefore support a regular representation of the orbifold group which usually corresponds to a bulk brane. This brane carries untwisted R-R charge and is in fact BPS (using the techniques of [32] one can check that the branes-at-angles configuration on the covering space is such that it preserves 1/4 supersymmetry), but does not carry any twisted R-R charge.

As we have explained before, the total action of the orbifold group on the open string space of states is given by (2.8). Consistency requires that this defines a conventional representation of the orbifold group. A general multi-point amplitude of some open string states can be factorised into the amplitude of the open string vertex operators, and a trace over the Chan-Paton indices. It is then conventional to define $U(g)$ on the different open string states so that the amplitude of the open string vertex operators is invariant under the group action. Since the total action has to be a conventional representation of the orbifold group, this fixes then the action of the orbifold group on the Chan-Paton indices.

Sometimes the action of $U(g)$ on the open string Hilbert spaces is canonically defined (such as in the case $N=2$ we discussed in section 2), but in general this may not be the case, in particular, if the group elements do not map the open strings to parallel strings. In such cases, the invariance of the amplitudes of the vertex operators fixes in principle the action of $U(g)$, but it is difficult to determine the precise action in practice.

One example for which this discussion is relevant is the bulk $D(r;1,1,1)$ brane we have just considered. In this case, the different open strings are not mapped into parallel strings under the action of the orbifold group, and there is therefore no canonical way to define the group elements on the open string space of states. It is therefore difficult to decide whether the action of the orbifold group on the Chan-Paton factors is a conventional or a projective representation in this case.

As we have argued above, the $D(r;1,1,1)$ brane is a bulk brane rather than a fractional brane. Another way to see this is to observe that it is impossible to write down a $D(r;1,1,1)$ boundary state in any twisted sector:[5] since N is odd, the oscillators in each twisted sector are not half-integer moded for at least two of the three complex directions $x^{2j+1}+ix^{2j+2}$. If this is the case, the only boundary condition that has a non-trivial solution is DD or NN (where the two letters refer to the boundary conditions for x^{2j+1} and x^{2j+2}, respectively.) On the other hand, the $D(r;1,1,1)$ brane has a mixed DN boundary condition for each of the three complex directions.

However, this argument does not exclude the existence of other fractional D-branes. In fact, the theory has a conventional fractional (non-BPS) D-brane for which exactly one s_i is odd. For instance, a brane with $s_1=1$ can have components

[5]We thank Fred Roose for a discussion on this point.

in the sectors twisted by g_1^m, because these group elements do not shift the modings in the complex direction for which we have a mixed (DN) boundary condition (x^3+ix^4). The boundary state corresponding to such a brane is

$$|D(r;s_1,s_2,s_3);\mathbf{y},\mathbf{a},\epsilon,\epsilon_1\rangle = \sum_{m=0}^{N-1}\epsilon_1^m \sum_{n=0}^{N-1} |D(r;s_1,s_2,s_3);\mathbf{y},g_2^n\mathbf{a},\epsilon\rangle_{\mathrm{T}_{g_1^m}}\,, \tag{4.2}$$

where it is understood that the orientations of the different component branes are such that the total configuration is orbifold invariant. Here we have used the short-hand notation

$$\begin{aligned} &|D(r;s_1,s_2,s_3);\mathbf{y},\mathbf{a},\epsilon\rangle_{\mathrm{T}_{g_1^m}} = \\ &\qquad = |D(r;s_1,s_2,s_3);\mathbf{y},\mathbf{a}\rangle_{\mathrm{NS-NS;T}_{g_1^m}} + \epsilon|D(r;s_1,s_2,s_3);\mathbf{y},\mathbf{a}\rangle_{\mathrm{R-R;T}_{g_1^m}}\,. \end{aligned} \tag{4.3}$$

For consistency with the group relations, ϵ_1 has to satisfy $\epsilon_1^N=1$. The associated $N\times N$ matrices defining the action on Chan-Paton factors are then

$$\gamma(g_1) = \mathrm{diag}(\epsilon_1,\epsilon_1,\ldots,\epsilon_1)\,; \quad \gamma(g_2) = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}\,. \tag{4.4}$$

This defines indeed a conventional representation of the orbifold group. The D-brane is non-supersymmetric — the branes-at-angles configuration in the covering space does not satisfy the criteria to preserve any supersymmetry, see for instance [32, 33] — and, in fact, unstable (as in the $N=2$ case, one can show that the open string spectrum contains tachyonic modes). These branes do not carry any R-R charges: although the boundary state has R-R components in the untwisted and the g_1^m-twisted sectors, there is no coupling to a massless R-R potential (it is projected out by the subgroup generated by g_2).

As before, we should check whether the various open strings carry consistent representations of the orbifold group and $(-1)^F$. Unfortunately, this is again very difficult to do explicitly since none of the orbifold generators maps any of the constituent branes to a parallel brane, and therefore, none of them can be represented by fermionic zero modes.

Apart from the bulk D$(r;1,1,1)$ branes for which the representation of the orbifold group is not easily determined, the branes that we have considered above all transform in a conventional representation of the orbifold group. This is to be contrasted with the situation for $N=2$ where the theory without discrete torsion also has branes that transform in a projective representation of the orbifold group. On the other hand, for those branes for which we can unambiguously identify the action of the orbifold group on the Chan-Paton indices (namely the D$(r;0,0,0)$ brane and the D$(r';1,0,0)$ brane) the results for $N=2$ and odd $N>2$ agree.

4.2 The theory with minimal discrete torsion

Like the $N = 2$ theory with discrete torsion, the theory has projective fractional D-branes where all s_i are even. These branes carry untwisted R-R charge and are BPS. The moduli space consists of three different branes, and the boundary state for the g_1 branch is given by

$$|D(r; s_1, s_2, s_3); \mathbf{y}, \mathbf{a}, \epsilon, \epsilon_1\rangle = \sum_{m=0}^{N-1} \epsilon_1^m \sum_{n=0}^{N-1} \omega^{-mn} |D(r; s_1, s_2, s_3); \mathbf{y}, g_2^n \mathbf{a}, \epsilon\rangle_{\mathrm{T}_{g_1^m}} , \quad (4.5)$$

where we have used (4.3) again. The powers of the discrete torsion phase ω are determined by the condition that the boundary state must be invariant under the action of the orbifold group.[6] The associated $N \times N$ matrices that define the action of the orbifold group on the Chan-Paton indices are now given as

$$\gamma(g_1) = \mathrm{diag}(\epsilon_1, \epsilon_1\omega^{-1}, \ldots, \epsilon_1\omega^{-(N-1)}) ; \quad \gamma(g_2) = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix} . \quad (4.6)$$

They define a projective representation of the orbifold group that is characterised by

$$\gamma(g_1)\gamma(g_2) = \omega^{-1}\gamma(g_2)\gamma(g_1) . \quad (4.7)$$

This is in agreement with what was found in [7, 12].

In addition to these projective fractional branes, the theory has also bulk D$(r; 1, 1, 1)$ branes, just as in the case without discrete torsion. The bulk branes also carry untwisted R-R charge and are BPS. As before, these branes cannot be fractional, and the corresponding representation on the Chan-Paton indices is not easily determined.

The above branes account for all of the untwisted R-R charges of the theory. However, the theory with discrete torsion also has twisted R-R charges [2, 7]. The branes that are charged with respect to these are non-BPS D-branes for which precisely one of the s_i is equal to 1 while the other two s_j are even. For the case $s_1 = 1$, the relevant boundary state takes the form

$$|D(r; s_1, s_2, s_3); \mathbf{y}, \mathbf{a}, \epsilon, \epsilon_1\rangle = \sum_{m=0}^{N-1} \epsilon_1^m \sum_{n=0}^{N-1} \omega^{-mn} |D(r; s_1, s_2, s_3); \mathbf{y}, g_2^n \mathbf{a}, \epsilon\rangle_{\mathrm{T}_{g_1^m}} . \quad (4.8)$$

[6] We are using here the convention that in the theory with discrete torsion, the action of g_i on the sector twisted by g_j is modified by multiplication with $\epsilon(g_i, g_j)$; see [26].

This leads to an open string with an $N \times N$ Chan-Paton matrix, for which the action of g_1 and g_2 is given again by (4.6).

This projective fractional non-BPS D-brane is quite an unusual object. It carries twisted R-R charge (unlike the situation in the theory without discrete torsion, the orbifold projection does not remove the relevant massless R-R potential to which it couples) which is *not* carried by any of the BPS branes of the theory that we have constructed above.[7]

Since none of the (known) BPS D-branes carry twisted R-R charge, one may expect that the non-BPS D-brane (that does carry this charge) has preferred stability properties, and this is indeed what we find.[8] As an example let us analyse the stability of the $\mathrm{D}(r; 1, 0, 0)$ brane. Before the orbifold projection, the open string between two different copies of the brane on the covering space has a tachyonic ground state. However, it follows from (4.6) that this ground state is projected out by g_1 because g_1 multiplies strings with off-diagonal Chan-Paton factors by a non-trivial phase. One may think that an orbifold invariant tachyonic state could be obtained by giving the open string momentum or winding. However, since g_1 acts trivially on the 34 directions, this must necessarily involve the 5678 directions. For the $\mathrm{D}(r; 1, 0, 0)$ brane, the open string has only winding modes in these directions, and they are infinitely massive in the uncompactified theory. Thus the $\mathrm{D}(r; 1, 0, 0)$ brane appears to be stable in the non-compact orbifold.[9] If we compactify, say, the 56 directions on a sufficiently small torus,[10] we get a tachyonic winding state that presumably signals the decay into a configuration of non-BPS $\mathrm{D}(r; 1, 2, 0)$ branes. Since the shape of the torus is fixed by the orbifold symmetry, there are no intermediate configurations; this is in agreement with the fact that none of the other branes that we have constructed carry twisted R-R charge.

In turn, a $\mathrm{D}(r; 1, 2, 0)$ brane is unstable in the uncompactified theory (because the open string has a tachyon with infinitesimal momentum in the 56 directions), but becomes stable if the 56 directions are compactified on a sufficiently small torus.

[7]Strictly speaking, we have not proven that these are the only BPS branes in the theory; on the basis of our analysis we can therefore not exclude the possibility that there are unknown BPS branes that carry this twisted charge. On the other hand, the relevant twisted R-R charge does not appear as a central charge in the supersymmetry algebra, and there is therefore no intrinsic reason why the BPS states should be charged under these potentials. Furthermore, examples of manifolds are known that have non-trivial two-cycles in homology, but for which no two-cycle can be chosen to be supersymmetric [34].

[8]The situation is in fact similar to what was found in [35]: the different non-BPS D-branes can decay into one another but do not seem to decay into BPS brane anti-brane pairs.

[9]The absence of a tachyonic mode does not imply in general that the D-brane is stable; for example, a D-brane whose open string does not contain a tachyonic mode may be metastable [36]. In the present case, however, there is no reason to suppose that the brane is only metastable.

[10]Strictly speaking, this only applies to $N = 3$ since we cannot compactify the orbifold for any other odd N.

5. The $\mathbb{Z}_N \times \mathbb{Z}_N$ orbifold for even N

5.1 The theory without discrete torsion

For N even, the construction of the conventional fractional BPS branes for which all three s_i are even and of the conventional fractional branes for which precisely one s_i is odd, is exactly the same as for odd N. What does change, however, is the situation for the D$(r;1,1,1)$ branes: a copy of such a brane on the covering space can now be mapped to itself by some elements of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ subgroup of $\mathbb{Z}_N \times \mathbb{Z}_N$. This opens up the possibility of having some kind of fractional D$(r;1,1,1)$ brane. It will turn out that the situation depends on whether N is divisible by four or not. In the following we shall write $N = 2M$; the situation will then depend on whether M is even or odd.

As in the case $N = 2$ we expect that the fractional D$(r;1,1,1)$ brane will have a moduli space with three different branches that are the fixed planes of g_1, g_2 and $g_3 = g_1 g_2$. For the first branch, a boundary state can be given by

$$\begin{aligned}
&|D(r;1,1,1);\mathbf{y},\mathbf{a},\epsilon,\epsilon'\rangle = \\
&= \sum_{m,n=0}^{M-1} \Big\{ |D(r;1,1,1);\mathbf{y},g_1^m g_2^n \mathbf{a}\rangle_{\mathrm{NS-NS;U}} + \epsilon |D(r;1,1,1);\mathbf{y},g_1^m g_2^n \mathbf{a}\rangle_{\mathrm{R-R;U}} + \\
&\qquad + \epsilon' \Big(|D(r;1,1,1);\mathbf{y},g_1^m g_2^n \mathbf{a}\rangle_{\mathrm{NS-NS;T}_{g_1^M}} + \epsilon |D(r;1,1,1);\mathbf{y},g_1^m g_2^n \mathbf{a}\rangle_{\mathrm{R-R;T}_{g_1^M}} \Big) + \\
&\qquad + |D(r;1,1,1);\mathbf{y},g_1^m g_2^n (-\mathbf{a})\rangle_{\mathrm{NS-NS;U}} + \epsilon |D(r;1,1,1);\mathbf{y},g_1^m g_2^n (-\mathbf{a})\rangle_{\mathrm{R-R;U}} - \\
&\qquad - \epsilon' \Big(|D(r;1,1,1);\mathbf{y},g_1^m g_2^n (-\mathbf{a})\rangle_{\mathrm{NS-NS;T}_{g_1^M}} + \\
&\qquad\qquad + \epsilon |D(r;1,1,1);\mathbf{y},g_1^m g_2^n (-\mathbf{a})\rangle_{\mathrm{R-R;T}_{g_1^M}} \Big) \Big\} ;
\end{aligned} \tag{5.1}$$

the construction for the other branches is analogous. We can read off from this boundary state that the representation of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ subgroup of $\mathbb{Z}_N \times \mathbb{Z}_N$ on the Chan-Paton indices is given by the direct sum of M^2 copies of (2.5) (the multiplicity of M^2 is due to the fact that M^2 different orientations are necessary to make an orbifold invariant configuration),

$$\begin{aligned}
\gamma(g_1^M) &= 1_{M\times M} \otimes 1_{M\times M} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\
\gamma(g_2^M) &= 1_{M\times M} \otimes 1_{M\times M} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ,
\end{aligned} \tag{5.2}$$

where we have chosen a particular sign in (2.5). This defines a projective representation of $\mathbb{Z}_2 \times \mathbb{Z}_2$,

$$\gamma(g_1^M)\gamma(g_2^M) = -\gamma(g_2^M)\gamma(g_1^M) \tag{5.3}$$

which is consistent with the projective $\mathbb{Z}_2 \times \mathbb{Z}_2$ action

$$g_1^M g_2^M = -g_2^M g_1^M \tag{5.4}$$

on the oscillator states of an open string between this brane and a conventional fractional D$(r';0,0,0)$ brane (see the discussion for $N = 2$ in section 2).

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Of course, we need to study the action of the whole $\mathbb{Z}_N \times \mathbb{Z}_N$ orbifold group on the open string states. From the discussion for the $\mathbb{Z}_2 \times \mathbb{Z}_2$ subgroup, it might seem natural to expect a projective representation of $\mathbb{Z}_N \times \mathbb{Z}_N$ on the oscillator states and on the Chan-Paton factors. However, as we shall now show, this can only be the case if M is odd, i.e. if N is not divisible by four. In order to see this, let us recall that a projective representation r satisfies

$$r(g_1)r(g_2) = \widehat{\omega}^{-1} r(g_2)r(g_1)\,. \tag{5.5}$$

Since $g_1^N = e$, it follows from (5.5) that

$$\widehat{\omega}^N = \widehat{\omega}^{2M} = 1\,. \tag{5.6}$$

Similarly, we can derive from (5.5) that

$$r(g_1^M)r(g_2^M) = \widehat{\omega}^{-M^2} r(g_2^M)r(g_1^M)\,, \tag{5.7}$$

so that consistency with (5.4) or (5.3) would require

$$\widehat{\omega}^{M^2} = -1\,. \tag{5.8}$$

It is then clear that (5.6) and (5.8) are only consistent if M is odd. Let us therefore study odd and even M separately.

For odd M, we can define the action of the orbifold generators on the $MN \times MN$ Chan-Paton factors to be given by

$$\gamma(g_1) = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}_{M\times M} \otimes \mathrm{diag}(1, \widehat{\omega}^{-1}, \ldots, \widehat{\omega}^{-(N-1)})_{N\times N}$$

$$\gamma(g_2) = \mathrm{diag}(1,1,\ldots,1)_{M\times M} \otimes \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}_{N\times N}\,, \tag{5.9}$$

where $\widehat{\omega}^M = -1$. This defines a projective representation of $\mathbb{Z}_N \times \mathbb{Z}_N$,

$$\gamma(g_1)\gamma(g_2) = \widehat{\omega}^{-1}\gamma(g_2)\gamma(g_1)\,, \tag{5.10}$$

and it defines a representation equivalent to (5.2) for the $\mathbb{Z}_2 \times \mathbb{Z}_2$ subgroup generated by g_1^M and g_2^M. This is sufficient to guarantee that the resulting open string satisfies the only easily testable consistency condition which comes from the action of g_1^M and g_2^M.

For even M, i.e. for N a multiple of four, it follows from (5.6) and (5.8) that the action of g_1 and g_2 on the oscillator states of the open string does not even define a representation that can be written as a direct sum of conventional and projective representations. (The action of g_1 and g_2 in the sector that describes the open strings between the $\mathrm{D}(r;0,0,0)$ and the $\mathrm{D}(r';1,1,1)$ brane is not even a projective representation.) In order for the total action to be consistent, this then implies that the same has to be the case for the action on the Chan-Paton indices. It is not difficult to see that one can choose an action on the Chan-Paton indices that reproduces (5.3), and that has the property

$$\gamma(g_1)\gamma(g_2) = \mathcal{R}\gamma(g_2)\gamma(g_1)\,, \tag{5.11}$$

where $\mathcal{R}$ is a diagonal matrix whose diagonal elements are ± 1. (If γ was a projective representation, $\mathcal{R} \propto \mathbf{1}$; the above is therefore a mild generalisation of a projective representation.) For example we can define the $MN \times MN$ matrix

$$\gamma(g_1) = \begin{pmatrix} P & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & P & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & P & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \widehat{P} & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & \widehat{P} & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \widehat{P} \end{pmatrix}, \tag{5.12}$$

where the first M matrices on the diagonal are

$$P = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}_{M\times M}\,, \tag{5.13}$$

while the second M matrices on the diagonal are

$$\widehat{P} = \begin{pmatrix} 0 & 0 & \cdots & 0 & -1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}_{M\times M}\,. \tag{5.14}$$

We also define $\gamma(g_2)$ to be the $MN \times MN$ matrix

$$\gamma(g_2) = \begin{pmatrix} 0 & 0 & \cdots & 0 & \mathbf{1}_M \\ \mathbf{1}_M & 0 & \cdots & 0 & 0 \\ 0 & \mathbf{1}_M & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & \mathbf{1}_M & 0 \end{pmatrix}, \tag{5.15}$$

and it is then not difficult to check that these matrices obey (5.3), and define a 'representation' satisfying (5.11) where $\mathcal{R}$ has ± 1 on the diagonal. This action on Chan-Paton factors can be combined with an action on the oscillator ground states of the open strings to give a conventional total action on the open string space of states. For example, for the open string between a fractional $\mathrm{D}(r;1,1,1)$ brane and a conventional fractional $\mathrm{D}(r';0,0,0)$ brane, we can choose the action on the ground states of the different open string sectors to be given by the above matrices. Then the total action in (2.8) is a conventional representation of the orbifold group. The action on the open string ground states so defined is consistent with the canonical action defined for g_1^M and g_2^M in terms of fermionic zero modes, as follows from the fact that the above matrices reproduce correctly the commutation relations for g_1^M and g_2^M.

5.2 The theory with minimal discrete torsion

As in the theory without discrete torsion, the only real difference with the analysis for odd N concerns the $\mathrm{D}(r;1,1,1)$ brane. Again, the analysis depends on whether $N=2M$ is a multiple of four, and we shall therefore consider the two cases (M odd and M even) separately.

For M odd, one may expect that the situation is quite similar to the case $M=1$ ($N=2$) that we discussed in section 2, and this is indeed true. The relevant boundary state is of the form

$$
\begin{aligned}
&|D(r;1,1,1);\mathbf{y},\epsilon,\epsilon_1,\epsilon_2\rangle = \qquad (5.16)\\
&= \sum_{m,n=0}^{M-1} g_1^m g_2^n \Big\{ |D(r;1,1,1);\mathbf{y}\rangle_{\mathrm{NS-NS;U}} + \epsilon |D(r;1,1,1);\mathbf{y}\rangle_{\mathrm{R-R;U}} + \\
&\qquad + \epsilon_1 \Big(|D(r;1,1,1);\mathbf{y}\rangle_{\mathrm{NS-NS;T}_{g_1^M}} + \epsilon |D(r;1,1,1);\mathbf{y}\rangle_{\mathrm{R-R;T}_{g_1^M}} \Big) + \\
&\qquad + \epsilon_2 \Big(|D(r;1,1,1);\mathbf{y}\rangle_{\mathrm{NS-NS;T}_{g_2^M}} + \epsilon |D(r;1,1,1);\mathbf{y}\rangle_{\mathrm{R-R;T}_{g_2^M}} \Big) + \\
&\qquad + \epsilon_1\epsilon_2 \Big(|D(r;1,1,1);\mathbf{y}\rangle_{\mathrm{NS-NS;T}_{(g_1g_2)^M}} + \epsilon |D(r;1,1,1);\mathbf{y}\rangle_{\mathrm{R-R;T}_{(g_1g_2)^M}} \Big) \Big\} ,
\end{aligned}
$$

where the action of the group elements includes discrete torsion phases. Since we are considering the case of minimal discrete torsion, $\omega^{M^2}=-1$, each of the boundary states is invariant under g_1^M and g_2^M. Together with the fact that we have summed together M^2 copies, this implies that the resulting boundary state is invariant under the action of the orbifold group. It is not difficult to see that it gives rise to a conventional action of the orbifold group on the Chan-Paton indices. The consistency of this conventional fractional $\mathrm{D}(r;1,1,1)$ brane can be tested as before, and the analysis is again analogous to the case $N=2$.

If M is even, i.e. if N is a multiple of four, $\omega^{M^2}=+1$, and therefore the boundary components of (5.16) in the twisted sectors are not invariant under the action of the orbifold group. In order to construct a non-trivial boundary state we therefore have to consider $NM(=2M^2)$ copies of the boundary states (rather than M^2); the relevant boundary state is then given as

$$
\begin{aligned}
&|D(r;1,1,1);\mathbf{y},\mathbf{a},\epsilon,\epsilon'\rangle = \qquad (5.17)\\
&= \sum_{m,n=0}^{M-1}\Big\{|D(r;1,1,1);\mathbf{y},g_1^m g_2^n\mathbf{a}\rangle_{\mathrm{NS-NS;U}} + \epsilon|D(r;1,1,1);\mathbf{y},g_1^m g_2^n\mathbf{a}\rangle_{\mathrm{R-R;U}} +\\
&\qquad + \epsilon'\omega^{-Mn}\Big(|D(r;1,1,1);\mathbf{y},g_1^m g_2^n\mathbf{a}\rangle_{\mathrm{NS-NS;T}_{g_1^M}} +\\
&\qquad\qquad + \epsilon|D(r;1,1,1);\mathbf{y},g_1^m g_2^n\mathbf{a}\rangle_{\mathrm{R-R;T}_{g_1^M}}\Big) +\\
&\qquad + |D(r;1,1,1);\mathbf{y},g_1^m g_2^n(-\mathbf{a})\rangle_{\mathrm{NS-NS;U}} + \epsilon|D(r;1,1,1);\mathbf{y},g_1^m g_2^n(-\mathbf{a})\rangle_{\mathrm{R-R;U}} -\\
&\qquad - \epsilon'\omega^{-Mn}\Big(|D(r;1,1,1);\mathbf{y},g_1^m g_2^n(-\mathbf{a})\rangle_{\mathrm{NS-NS;T}_{g_1^M}} +\\
&\qquad\qquad + \epsilon|D(r;1,1,1);\mathbf{y},g_1^m g_2^n(-\mathbf{a})\rangle_{\mathrm{R-R;T}_{g_1^M}}\Big)\Big\}.
\end{aligned}
$$

It is not difficult to check that this boundary state is invariant under the action of the orbifold group.

As before for the case without discrete torsion, the action on the Chan-Paton factors that follows from this boundary state does not even define a projective representation, but only satisfies (5.11). In fact we have again that $\gamma(g_1^M)\gamma(g_2^M) = -\gamma(g_2^M)\gamma(g_1^M)$, and therefore this is necessarily so. However, it can be checked as in the theory without discrete torsion (and with similar limitations) that this brane is consistent with the other branes of the theory. For example, the open string to the D$(r';0,0,0)$ brane is consistent since the above sign is cancelled by the sign appearing in (2.7), and the action of g_1^M and g_2^M on the Chan-Paton indices of the D$(r';0,0,0)$ brane commute since g_1 and g_2 act by a projective representation (with $\omega^{M^2}=+1$). The other cases are similar.

6. Conclusions

In this paper we have analysed the D-brane spectrum of orbifold theories with and without discrete torsion. We have carefully examined the consistency of the orbifold action on the open string states that correspond to the different open strings between the various D-branes. Already for the simplest interesting example, $\mathbb{C}^3/\mathbb{Z}_2\times\mathbb{Z}_2$, we have found that the analysis falls outside the scope of the framework discussed in [23]; in fact, the consistency requires that the D-brane spectrum contains branes that carry a conventional representation of the orbifold group as well as branes for which the representation is projective. This is precisely in agreement with the D-brane spectrum that had been proposed in [26] using the boundary state formalism and the constraints of T-duality. We have also analysed the non-BPS branes for this theory, for which additional subtleties arise. In particular, the consistency of the various symmetry operators requires that $(-1)^F$ acts non-trivially on the Chan-Paton indices of some non-BPS branes. As before, this is beautifully reproduced by the corresponding boundary states that we construct.

We have also analysed how these results generalise to the orbifolds $\mathbb{C}^3/\mathbb{Z}_N \times \mathbb{Z}_N$. Among other things we have found that some of the these theories have non-BPS D-branes carrying R-R charges that are not carried by any of the (known) BPS branes of the theory. These non-BPS D-branes enjoy special stability properties.

Most of our analysis has been done case by case, and it would be interesting to be able to understand these results more conceptually. In particular, it should be possible to understand the nature of the representation of the orbifold group on the Chan-Paton indices of a given brane more abstractly, for example in terms of K-theory. The consistency of the various open string actions should then also follow from some abstract arguments.

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